Non–trivial Fixed Point in Four Dimensional Scalar Field Theory and the Higgs Mass

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We show, using the large N limit, that there is a non–trivial scale invariant action for four dimensional scalar field theory. We investigate the possibility that the scalar sector of the standard model of particle physics has such a scale invariant action, with scale invariance being spontaneously broken by the vacuum expectation value of the scalar. This leads to a prediction for the mass of the lightest massive scalar particle (the Higgs particle) to be 5.4 Tev.

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We will study the possibility of a non–trivial ultra–violet stable fixed point in a four dimensional scalar field theory. We will show, to leading order in the $\frac{1}{N}$ expansion,[1] that such a fixed point does exist. A non–trivial fixed point in three dimensional scalar field theory was found by Wilson and Fisher; it describes second order phase transitions. These classic studies of renormalization did not find a fixed point in the four dimensional case[2]. We studied the three dimensional case by the large N method, constructing the corresponding three dimensional conformal field theory in leading order of the $\frac{1}{N}$ expansion[3]. There are theorems that constrain the existence of a fixed point in four dimensional scalar theory [4]. These analyses are rigorous and non-perturbative, and our results do not contradict them. The triviality arguments apply to $\lambda\phi^4$ type of interactions. The self–interactions of the theory we describe are not of this type. The interaction potential is not a polynomial, or even a real analytic function, of the field ϕ. It is a function of ϕ^2 (ϕ being the scalar field) with a logarithmic branch point at $\phi = 0$ and grows like $\frac{\phi^4}{\log \phi^2}$ for large ϕ^2.(The form of the potential depends on the regularization scheme used, and we just described the situation in zeta function regularization. See below for details.) Such interactions are not treatable within perturbation theory, indeed the potential has no power series expansion in the field ϕ^2; they are not included in the studies that find a negative result on non–trivial fixed points.

Our result has an application to the standard model[5]. The definition of the standard model by perturbative renormalization is 'unnatural' [6] in the following sense : the values of the parameters in the scalar sector (in paricular the Higgs mass) depend very sensitively on the short distance (microscopic) behaviour of the theory. We can now avoid this difficulty by changing just the scalar self–interaction, leaving all other aspects of the standard model intact. Recall that this part of the standard model has not yet been tested[6].

The only place in the standard model Lagrangian where scale invariance is explicitly broken is in the scalar self–interaction. Perturbative renormalizability requires the scalar self-interaction to be a quartic polynomial. We propose that the standard model be modified by choosing the scalar self–interaction such that the Lagrangian is scale invariant even after including quantum corrections. This can be viewed as a non–perturbative renormalizability condition. This fixes the potential, but the answer is not a polynomial or even an analytic function of the field. Scale invariance would then be spon-

taneously broken by the vacuum expectation value of the scalar field, just like the internal symmetry. Then it would be possible to determine the masses of the scalar particles as multiples of this vacuum expectation value. We will determine this fixed point potential, to leading order in the $\frac{1}{N}$ expansion. In a further approximation, we will calculate the mass of the lightest massive scalar.

Let us return to the study of scalar field theory. We will consider a real scalar field ϕ_i with $N+1$ components, $i = 0, 1, 2, \cdots N$ and analyze it in the $\frac{1}{N}$ expansion. The Lagrangian will be required to have a global symmetry under the group $O(N+1)$. Upto some complications of regularization (see below),

$$L_1 = \frac{1}{2}\Big[|\partial\phi_i|^2 + NW(\frac{\phi^2}{N})\Big]. \tag{1}$$

The factors of N have been put in for later convenience. The self–interactions of the scalar field are contained in the potential W; it is usually taken to be a quadratic in ϕ^2. We will instead determine it by the requirement that the Lagrangian be scale invariant even after including the effects of fluctuations in the field ϕ_i.

This Lagrangian can be viewed as describing the scalar sector of the standard model, in the limit where all gauge and Yukawa interactions are ignored. The scalar field then has four real components, and the global symmetry of the scalar sector is $O(4)$.

It will be convenient to consider an equivalent Lagrangian with an auxilliary field σ,

$$L_2 = \frac{1}{2}[|\partial\phi_i|^2 + N[\sigma\frac{\phi_i^2}{N} - V(\sigma)]] \tag{2}$$

where W and V are related by the one–dimensional integral:

$$\int_0^\infty e^{-N[\sigma\eta - V(\sigma)]}d\sigma = e^{-NW(\eta)} \tag{3}$$

where $\eta = \frac{\phi^2}{N}$. In the large N limit V and W will be Legendre transforms of each other. We will now integrate over all except one of the fields ϕ_i leaving σ and $\phi_0 = \sqrt{N}b$. The resulting effective action will be

$$S[b,\sigma] = -\ln\int D[\phi]e^{-\int L_2[\phi,b,\sigma]d^4x} \tag{4}$$

which is

$$S[b,\sigma] = N\frac{1}{2}\int [|\nabla b|^2 d^4x + \sigma b^2 - V(\sigma)]d^4x + \frac{1}{2}N \text{ Tr } \ln[-\nabla^2+\sigma]. \quad (5)$$

Here, we have to confront issue of the definition of the divergent quantity $\text{Tr } \ln[-\nabla^2+\sigma]$. We will define it by the zeta function method which is technically simple and elegant. It is possible to use other regularizations such as momentum cutoff; the details tend to be more complicated however.

The idea of the zeta function method is that, for $\sigma > 0$, the quantity

$$\zeta(s,x) =< x|[-\nabla^2+\sigma]^{-s}|x > \quad (6)$$

is a regular analytic function of s for large enough Re s. It therefore defines an analytic function on the complex s–plane, which in our case has only simple pole singularities, located at $s = 1, 2$. In particular, this function is regular analytic at $s = 0$, so that we can define ($'$ denotes differentiation with respect to s)

$$\text{Tr } \ln[-\nabla^2+\sigma] = -\int \zeta'(0,x)d^4x. \quad (7)$$

For finite dimensional operators this agrees with the usual definition of the Tr ln.

Although this gives a definition of the Tr ln without any apparent scale (such as a momentum cutoff), the result is in fact *not* scale invariant: there is an anomaly. We can see this most easily by calculating it for a constant background $\sigma = m^2$:

$$\zeta(s) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{[p^2+m^2]^s} = \frac{1}{2}\frac{2\pi^2}{(2\pi)^4}\frac{m^{4-2s}}{(s-2)(s-1)} \quad (8)$$

so that by our definition,

$$\int \frac{d^4p}{(2\pi)^4}\ln[p^2+m^2] = \frac{1}{4}\frac{2\pi^2}{(2\pi)^4}m^4\ln[e^{-3/2}m^2]. \quad (9)$$

Clearly this is not scale invariant, due to the logarithmic term.

Now we propose that $V(\sigma)$ be chosen so as to cancel the vacuum energy of quantum fluctuations $\text{Tr } \ln[-\nabla^2+\sigma]$ around a constant background σ. To leading order in $\frac{1}{N}$,

$$V(\sigma) = \frac{1}{32\pi^2}\sigma^2\ln[e^{-3/2}\sigma] \quad (10)$$

Then the action $S[b,\sigma]$ is scale invariant under the transformations

$$b \to \lambda b \quad \sigma \to \lambda^2 \sigma \quad x \to \lambda^{-1} x. \tag{11}$$

This can be verified by computing the scale anomaly of $\mathrm{Tr}\ \ln[-\nabla^2 + \sigma]$ by heat equation methods and noting that it is cancelled by the variation of the potential. We will give detailed proofs in a longer paper.

In the exact theory, $V(\sigma)$ will still be determined by the condition of cancellation of the energy of constant backgrounds, but it will have corrections of order $\frac{1}{N}$. It will still be scale invariant, each contribution to the scale anomaly will be cancelled by a variation of a term in $V(\sigma)$. This nontrivial fixed point of the renormalization group can thus be constructed order by order in the $\frac{1}{N}$ expansion.

It is possible to generalize the action to curved spaces preserving conformal invariance. $S[b,\sigma]$ defines a conformal field theory, although with a nonlocal action. The three dimensional analogue of this was studied in [3]. There is however, no trace anomaly in three dimensions, and there are no logarithimc terms in the potential $V(\sigma)$.

The form of the potential that describes the fixed point depends on the regularization scheme. Let us consider a momentum cut–off scheme as well, to make comparison with other approaches easier. If we choose a cut–off function K on the positive real line(which is one in a neighborhood of the origin and zero in a neighborhood of infinity), we get

$$V_K(\sigma) = \frac{1}{8\pi^2}\int_0^\infty \ln[p^2+\sigma] K(\frac{|p|}{\Lambda}) p^3 dp. \tag{12}$$

The difference $V_{K_1} - V_{K_2}$ between two choices K_1 and K_2 will be an analytic function of σ in the neighborhood of $\sigma = 0$. This can be seen by using the Lebesgue dominated convergence theorem, since $K_1 - K_2$ is non–zero only in some interval $0 < a_1 < \frac{|p|}{\Lambda} < a_2$. Thus the 'analytic germ' of V_K (i.e., its equivalence class under the addition of a function analytic at $\sigma = 0$) is independent of the choice of the cut–off function K. If we choose the step function, $K(\frac{|p|}{\Lambda}) = \theta(0 \le |p| < \Lambda)$ we get (putting $\Lambda = 1$ for simplicity),

$$V_K[\sigma] = \frac{1}{32\pi^2}\left[\sigma^2 \ln[\frac{\sigma}{\sqrt{e}}] + 2\sigma\right] + O(\sigma^3) \tag{13}$$

This has the same analytic germ as the answer we got in zeta function regularization.

We can determine the potential W once we know V: to leading order it is just the Legendre transform

$$W(\eta) = \max_{\sigma}[\eta\sigma - V(\sigma)] \tag{14}$$

of V. The form of W also depends on the regularization scheme. Consider first the form of the potential for zeta function regularization. Then the extremum occurs at

$$\eta = \frac{1}{16\pi^2}\sigma \ln[\frac{\sigma}{e}] \tag{15}$$

which can be solved recursively for $\eta > 0$:

$$\sigma(\eta) = \frac{16\pi^2\eta}{\ln[\frac{\sigma(\eta)}{e}]} \tag{16}$$

Once σ is determined as a function of η this way,

$$W[\eta] = \frac{1}{2}\sigma(\eta)[\frac{\sigma(\eta)}{32\pi^2} + \eta]. \tag{17}$$

This is not an analytic funtion of η at $\eta = 0$: the function $\sigma(\eta)$ has a logarithmic branchpoint at $\eta = 0$. For large η, we see that

$$W[\eta] \sim 8\pi^2\frac{\eta^2}{\log\eta}. \tag{18}$$

The non–analytic behaviour of $W[\eta]$ is present in other regularization schemes as well; the position of the logarithmic singularity however depends on the scheme used. If we take as an example

$$V(\sigma) = \frac{1}{32\pi^2}\sigma^2 \ln\frac{\sigma}{c_1\sqrt{e}} + c_2\sigma \tag{19}$$

the logarithmic branch point is at $\eta = c_2$. For $\eta < c_2$, the function $W[\eta]$ is multivalued, there are two extrema:if we follow the smaller of the two branches, it will vanish identically for $\eta < \eta_1 = c_2 - \frac{c_1}{16\pi^2 e}$. Thus the kind of potentials we consider cannot be studied in an expansion in powers of the field variable $\eta = \frac{\phi^2}{N}$; the interaction will look trivial to all orders of perturbation theory in the momentum cutoff regularization, for example. From now on we will only use zeta function regularization.

Although the action S is scale invariant, it is possible to have extrema that break scale invariance. With the above choice of potential, for constant backgrounds, the problem of extremizing S reduces to that of extemizing

$$\frac{1}{2}m^2 b_0^2 \tag{20}$$

where $\sigma = m^2$. There are two possible solutions: $b_0 = 0$ and m arbitrary or $m = 0$ and b_0 arbitrary. In the first case, the $O(N+1)$ symmetry is unbroken and in the second it can be spontaneously broken. Let us now study the spectrum of the oscillations around this constant background. This is a subtle problem due to various Infrared divergences; we will only attempt to get an estimate of the mass of the lightest massive paricle in the scalar sector.

The Weyl calculus[7] can be used to obtain an expansion for the Tr ln in powers of the derivatives of σ. It will be convenient to put $\sigma = \chi^2$. We will study only the longest wavelength fluctuations in σ, so we will only calculate the leading term in the expansion of $\mathrm{Tr}\ \ln[-\nabla^2 + \sigma] - \int V(\sigma)dx$ in terms of the derivatives of σ. We will see that this term has a simple form in terms of derivatives of χ, so that χ rather than σ is the natural variable to use. We will give details of this rather long calculation in another paper, noting here just that the leading order terms in the expansion for the zeta function are,

$$\zeta(s) = \frac{1}{16\pi^2}[\frac{\sigma^{2-s}}{(s-1)(s-2)} - \frac{1}{6}\sigma_{ii}\sigma^{-s} + \frac{1}{12}s\sigma^{-s}\frac{\sigma_i^2}{\sigma}] + \cdots. \tag{21}$$

Now we have the expansion for the Tr ln,

$$\mathrm{Tr}\ \ln[-\nabla^2 + \sigma] = \frac{1}{32\pi^2}\int\left[\sigma^2 \ln(e^{-\frac{3}{2}}\sigma) + \frac{1}{6}\frac{\sigma_i^2}{\sigma}\right]dx + \cdots. \tag{22}$$

Now put $\sigma = \chi^2$ to get

$$\mathrm{Tr}\ \ln[-\nabla^2 + \chi^2] = \frac{1}{32\pi^2}\int\left[\chi^4 \ln(e^{-\frac{3}{2}}\chi^2) + \frac{2}{3}(\nabla\chi)^2\right]dx + \cdots \tag{23}$$

upto higher derivatives of χ.

We can combine this with our original expression for the action,to get,

$$S[b,\chi] = \frac{1}{2}\int[|\nabla b|^2 + \chi^2 b^2]dx + \frac{1}{2}\frac{2}{3}\frac{1}{32\pi^2}\int|\nabla\chi|^2 dx + \cdots \tag{24}$$

The term with no derivatives in χ is cancelled by our choice of V. Therefore, for any constant b_0, $b = b_0, \chi = 0$ is an extremum. The oscillations of χ will then have a mass

$$m_\chi = 4\pi\sqrt{3}b_0 \sim 21.7b_0. \tag{25}$$

There could be other modes of oscillation ('particles') as well, but the approximation we use should apply only for the lightest massive particle.

Now we consider the possibility that the scale invariant potential W determined as above is the scalar self–interaction of the standard model. The standard model would then be 'natural', there being an ultraviolet stable fixed point in the scalar interactions. Since the only source of a scale in the scalar sector is then the vacuum expectation value, we can predict in principle the spectrum of scalar particles. This is a difficult problem, comparable to that of determining the spectrum of Quantum Chromodynamics. It is a problem for which lattice studies of the sort already carried[8] out for $\lambda\phi^4$ would be worthwhile. Here, we make only an estimate of the mass of the lightest particle in the scalar sector. This mass thus describes the longest wavelength oscillations in the magnitude of the scalar field. Although there may be several particles (and resonances) in the scalar sector, this lightest massive particle is the closest thing to a Higgs particle we would have in our scale invariant theory. With $b_0 \sim 250$ GeV (known from gauge boson masses) we get for this mass:

$$m_\chi \sim 5.4 \text{ Tev}. \tag{26}$$

An interesting consequence of our analysis is that there is no coupling of the form χb^2 in the action. The reflection symmetry $\chi \to -\chi$ is not spontaneously broken, so there will be no decay of this particle χ within the scalar sector. Thus within our approximations this particle is stable.

We emphasize that since ours is a non–perturbative approach to the scalar self–interactions, the usual perturbative unitarity bound[6] does not apply. There have been several other ideas on fixed points involving scalars [9],[10]. Our approach only affects the scalar self–interactions and does not introduce any new parameter in the standard model. We will address generalizations and higher order computations of this theory in future publications.

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